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Spinning and rotating strings for $\mathcal{N} = 1$ SYM theory and brane constructions

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Abstract: We obtain spinning and rotating closed string solutions in $AdS_5 \times T^{1,1}$ background, and show how these solutions can be mapped onto rotating closed strings embedded in configurations of intersecting branes in type IIA string theory. Then, we discuss spinning closed string solutions in the UV limit of the Klebanov-Tseytlin background, and also properties of classical solutions in the related intersecting brane constructions in the UV limit. We comment on extensions of this analysis to the deformed conifold background, and in the corresponding intersecting brane construction, as well as its relation to the deep IR limit of the Klebanov-Strassler solution. We briefly discuss on the relation between type IIA brane constructions and their related M-theory descriptions, and how solitonic solutions are related in both descriptions.

Keywords: Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

The large N limit of SYM theories is related to supergravity solutions through AdS/CFT duality [1]. Particularly, the first and best understood example of this phenomenon [2] is the duality between the large N limit of $\mathcal{N} = 4$ SU(N) SYM theory in 4d and type IIB supergravity on $AdS_5 \times S^5$ with N units of F_5 flux through S^5, being this background the near horizon limit of a configuration of N parallel D3-branes. Essentially, the system is considered in the decoupling limit, i.e. modes propagating on the brane world-volume and modes propagating in the bulk are decoupled.

More recent studies have shown that it is possible to extend these ideas beyond the supergravity approximation. In particular, certain $\mathcal{N} = 4$ SU(N) SYM theory operators with large R-symmetry charge have been proposed to be dual to certain closed string theory states [3], which are obtained by quantization of type IIB string theory on a pp-wave background [4]. The same conclusion has been reached by quantizing string theory around certain classical solutions, leading to semi-classical limits where the string/gauge theory duality can be extended [5]. In particular, short spinning closed strings in AdS_5 have been shown to reproduce Regge behavior. On the other hand, long spinning closed string solutions in AdS_5 reproduce the logarithmic anomalous scaling dimension for twist two SYM theory operators [5]–[9]. The last is a very important feature of the UV behavior of asymptotically free Yang-Mills theories both, supersymmetric and non-supersymmetric, and it is believed to be an universal property [10, 7].

The finding of new closed string solutions [6]–[9], [11]–[14] and [16]–[23] has motivated to explore different regions of the spectrum by using both sides, i.e. weakly coupled field theory and semi-classical string solutions. In addition, rotating closed membranes have

JHEP02(2004)066

been recently studied [19] and [24]–[26], obtaining generalizations of the energy-spin relations for spinning closed strings, including the logarithmic behaviour and energy-charge relations as well.

The interest in studying non-conformal SYM theories with less supercharges is obvious. In fact, during the last few years, a very extensive work has been done in order to develop the supergravity duals of field theories in the large N limit. For instance, the large N limit of $\mathcal{N}=2$ $\mathrm{SU}(N)\times\mathrm{SU}(N)$ SYM theory in 4d has been proposed to be dual to type IIB supergravity on $AdS_5\times S^5/Z_2$ [27]. From the viewpoint of $\mathcal{N}=1$ supersymmetry, the $\mathcal{N}=2$ theory contains two pairs of chiral bifundamentals and two adjoints, coupled by a cubic superpotential. After adding a mass perturbation and integrating out the adjoints, the superpotential of $\mathcal{N}=1$ $\mathrm{SU}(N)\times\mathrm{SU}(N)$ SYM theory in 4d is obtained. In this resulting $\mathcal{N}=1$ theory there are chiral superfields A_k, $k=1,2$ transforming in the $(\mathrm{N},\overline{\mathrm{N}})$ representation and B_l, $l=1,2$ transforming in the $(\overline{\mathrm{N}},\mathrm{N})$ representation, plus a quartic superpotential. This theory turns out to be dual to type IIB supergravity on $AdS_5\times T^{1,1}$ and N units of F_5 flux through $T^{1,1}$ [28]. D3-branes wrapped over the 3-cycles of $T^{1,1}$ can be identified with baryon-like chiral operators built out of products of N chiral superfields, while a D5-brane wrapped over a 2-cycle of $T^{1,1}$ acts as a domain wall in AdS_5 (usually called fractional D3-brane) [29]. Moreover, the dual description of N regular plus M fractional D3-branes at the conifold singularity is a non-conformal $\mathcal{N}=1$ $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ SYM theory. This supergravity dual description reproduces the logarithmic flow of couplings found in the field theory [30]. In reference [30] the leading order M/N effects were considered. Furthermore, the back-reaction of the fractional branes on the gravitational background was studied in [31], obtaining the logarithmic RG flow of couplings found in the field theory at all scales. In addition, Klebanov and Strassler showed how the gauge theory undergoes repeated Seiberg-duality transformations, leading to a reduction of the rank of both gauge groups in M units every time. They also noted that the gauge theory confines in the IR and, its chiral symmetry breaking removes the singularity of the Klebanov-Tseytlin solution by deforming the conifold [32]. More recent investigations include massive flavored fundamental quarks in the supergravity dual of $\mathcal{N}=1$ SYM theory by introducing D7-brane probes to the Klebanov-Strassler solution, finding a discrete spectra exhibiting a mass gap of the order of the glueball mass [34][1]. On the other hand, inspired in [5], semi-classical solutions of rotating closed strings in both, AdS_5 and $T^{1,1}$ spaces have been studied [7]. In particular, in ref.[20] classical string solutions were considered in the Klebanov-Tseytlin gauge theory compactified on S^3 [33]. In ref.[21] studies related to the Klebanov-Strassler background have been carried out. However, it has been shown that using Poincaré-like coordinates it is not possible to reproduce the well-known field theory relation between energy and angular momentum. The problems that one must face when studying near conformal backgrounds are of two different types [7]. One concerns to the fact that near conformal backgrounds are written in terms of Poincaré-like coordinates, while rotating string solutions usually become simpler as written in global coordinates. The second problem is related to the definition of the string theory analog of a conformal dimension for near conformal backgrounds.

[1]It is also very interesting the computation of the meson spectrum of an $\mathcal{N}=2$ SYM theory with fundamental matter from its dual string theory on $AdS_5\times S^5$ with a D7 brane probe done in ref.[35].

JHEP02(2004)066

A complementary viewpoint in the context of type IIA string theory can be developed under the observation that there is a map from certain type IIA intersecting brane configurations onto the conifold. In fact, time ago Bershadsky, Sadov and Vafa argued that the conifold singularity is dual to a system of NS fivebranes intersecting over a 3+1 dimensional worldvolume [36]. Then, Dasgupta and Muhki [37, 38] have shown that a set of parallel D3-brane probes near a conifold singularity can be mapped onto a configuration of intersecting branes in type IIA string theory. In this formulation they explicitly derived the field theory on the probes. Moreover, it is possible to show that brane constructions in type IIA string theory are related to large classes of chiral and non-chiral $\mathcal{N}=1$ field theories with quartic superpotentials [40]. It would be of interest to have an explicit realization for a system of type IIA intersecting branes related to the deep IR limit of the Klebanov-Strassler solutions. In addition, a deformation of the conifold leads to a smoothed out intersection of two sets of NS fivebranes [41]. Also, large N dualities for a general class of $\mathcal{N}=1$ theories on type IIB D5-branes wrapping 2-cycles of local Calabi-Yau three-folds, or as seen as effective field theories on D4-branes in type IIA brane configurations have been studied [42].

In this paper we investigate classical solutions in this complementary framework leading to a new perspective about the dual description of $\mathcal{N}=1$ SYM theories in 4d. We shall show how spinning and rotating closed string solutions in certain type IIB string theory backgrounds, which have been proposed to be dual to $\mathcal{N}=1$ SYM theories, can be mapped onto classical string solutions embedded in backgrounds induced by configurations of intersecting branes in type IIA string theory. In the case of the conifold, for long string solutions corresponding to large spin operators in the ultraviolet limit of SYM theory, we found the expected logarithmic anomalous dimension of minimal twist operators. In addition, infrared Regge behavior for spinning short strings is recovered. In the type IIA intersecting brane configurations studied here, there can also be non-vanishing B-fields. We shall therefore study this effect that provides new features of $\mathcal{N}=1$ SYM theory spectrum from the brane construction perspective. Different solitonic string solutions with B-fields were previously considered in [17, 18].

In section 2 we review some results on classical solutions for closed strings in type IIB string theory in conformal and near-conformal backgrounds. In section 3 we study the Dasgupta-Muhki map relating intersecting branes constructions to the conifold. Then, in section 3.2 we obtain classical string solutions in $AdS_5 \times T^{1,1}$ background, and we recover the expected features for long and short spinning closed strings. We map the above solutions onto classical solutions of rotating closed strings embedded in type IIA intersecting brane constructions. In section 3.3 we study the specific case of spinning closed strings in AdS_5 and rotating in S^5. We recover the expected features for long and short spinning closed strings. We also analyze the corresponding classical string solutions in the intersecting brane constructions. In section 4 we discuss about extensions of the map for the embedding of closed string solutions in the UV limit of the Klebanov-Tseytlin background. Then, using the Ohta-Yokono map we address similar issues for the case of the deformed conifold. Inspired in this, we also comment on classical solutions in the deep IR limit of the Kebanov-Strassler background. Although we have not completely succeeded in obtaining explicit relations of the form $E = E(S,J)$ for some of the string solutions presented in this paper, we think this work can motivate to explore more exhaustively this kind of relations in both, conifold-like and intersecting brane backgrounds, and study the corresponding semi-classical quantization, as well as string theory states/SYM theory operators correspondence. The last section 5 is devoted to a general discussion, also describing interesting open questions related to $\mathcal{N}=1$ SYM theory operators, and M-theory descriptions of the above mentioned systems.

2. Rotating and spinning closed string solutions in type IIB string theory in conformal and near-conformal backgrounds

In this section we review some results of rotating and spinning solutions for closed strings in type IIB string theory in conformal backgrounds [5]–[7], that can be extended to near-conformal ones [7]. Particularly, we review the case of boosted and spinning strings in $AdS_5 \times S^5$ background. One of the reasons of interest in this background is that for long strings the anomalous scaling dimension of twist two operators behaves like $E = S + f(\lambda)\,\ln S/\sqrt{\lambda} + \cdots$, where $\sqrt{\lambda} = R^2/\alpha'$, being R the AdS_5 radius. We leave for section 3 a more extensive study of classical solutions of $AdS_5 \times T^{1,1}$, and for section 4 a discussion on the solutions related to non-conformal $\mathcal{N}=1$ SYM theories, whose supergravity duals are the Klebanov-Tseytlin and Klebanov-Strassler backgrounds.

Let us consider the bosonic part of the Green-Schwarz superstring action in $AdS_5 \times S^5$

$$I_B = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma L_B(x)\,, \tag{2.1}$$

where

$$L_B(x) = \frac{1}{2}\sqrt{-g}g^{ab}G_{\mu\nu}\partial_a x^\mu \partial_b x^\nu\,. \tag{2.2}$$

We use Minkowski signature in both, the target space and world-sheet. In the conformal gauge we have $\sqrt{-g}g^{ab} = \eta^{ab} = diag(-1,1)$. In global coordinates the AdS_5 space-time metric can be written as

$$\begin{aligned} ds^2_{AdS_5} &= R^2[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3]\,,\\ d\Omega_3 &= d\beta_1^2 + \cos^2\beta_1(d\beta_2^2 + \cos^2\beta_2 d\beta_3^2)\,, \end{aligned} \tag{2.3}$$

while the parametrization of S^5 is

$$\begin{aligned} ds^2_{S^5} &= R^2[d\psi_1^2 + \cos^2\psi_1(d\psi_2^2 + \cos^2\psi_2 d\Omega'_3)]\,,\\ d\Omega'_3 &= d\psi_3^2 + \cos^2\psi_3(d\psi_4^2 + \cos^2\psi_4 d\psi_5^2)\,. \end{aligned} \tag{2.4}$$

Then, for a closed spinning string in $\phi \equiv \beta_3$ of AdS_5 and boosted along the direction $\varphi \equiv \psi_5$ of S^5, one can write the ansatz

$$\begin{aligned} t &= \kappa\tau\,, \qquad \phi = \omega\tau\,, \qquad \varphi = \nu\tau\,,\\ \rho(\sigma) &= \rho(\sigma + 2\pi)\,, \qquad \beta_i = 0 \quad (i=1,2)\,, \psi_j = 0 \quad (j = 1,\ldots,4)\,, \end{aligned} \tag{2.5}$$

where κ, ω and ν are constants. ρ is subject to the following second order equation

$$\rho'' = (\kappa^2 - \omega^2) \cosh\rho \sinh\rho \,, \tag{2.6}$$

which is implied by eq. (2.8) below, where prime stands for derivative with respect to σ. The conformal constraints are

$$G_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} = 0 \,, \qquad G_{\mu\nu} \left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} + \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \sigma} \right) = 0 \,. \tag{2.7}$$

The first equation is automatically satisfied for the above string configuration. Using the second conformal constraint and the previous ansatz for the string solution, it is derived the following relation

$$(\rho')^2 = \kappa^2 \cosh^2\rho - \omega^2 \sinh^2\rho - \nu^2 \,. \tag{2.8}$$

From eq. (2.8) one straightforwardly obtains

$$2\pi = \int_0^{2\pi} d\sigma = 4 \int_0^{\rho_0} d\rho \frac{1}{\sqrt{(\kappa^2 - \nu^2)\cosh^2\rho - (\omega^2 - \nu^2)\sinh^2\rho}} \,. \tag{2.9}$$

There are three conserved quantities, i.e. energy, spin and R-symmetry charge J,

$$E = \sqrt{\lambda}\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho \equiv \sqrt{\lambda}\mathcal{E} \,, \tag{2.10}$$

$$S = \sqrt{\lambda}\omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2\rho \equiv \sqrt{\lambda}\mathcal{S} \,, \tag{2.11}$$

$$J = \sqrt{\lambda}\nu \int_0^{2\pi} \frac{d\sigma}{2\pi} \equiv \sqrt{\lambda}\nu \,. \tag{2.12}$$

Thus, one obtains the relation

$$E = \frac{\kappa}{\nu} J + \frac{\kappa}{\omega} S \,. \tag{2.13}$$

At this point, one can explicitly calculate the above integrals in terms of hyper-geometric functions. It is useful to write down the energy-spin relations for both, short and long strings. In doing this, it is convenient to introduce a parameter $\eta > 0$, defined as

$$\coth^2\rho_0 = \frac{\omega^2 - \nu^2}{\kappa^2 - \nu^2} = 1 + \eta \,. \tag{2.14}$$

In fact, for short strings eq. (2.13) becomes

$$\mathcal{E} \approx \sqrt{\nu^2 + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}}} + \frac{\sqrt{\nu^2 + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}}}}{\sqrt{1 + \nu^2 + \frac{2\mathcal{S}}{\sqrt{1+\nu^2}}}} \mathcal{S} \,. \tag{2.15}$$

The above expression is valid when $\frac{1}{\eta} \approx \frac{2\mathcal{S}}{\sqrt{1+\nu^2}} \ll 1$. Now, if we demand $\nu \ll 1$, then $\mathcal{S} \ll 1$, and therefore eq. (2.15) reduces to

$$E^2 \approx J^2 + 2\sqrt{\lambda} S \,. \tag{2.16}$$

JHEP02(2004)066

This expression is the limit for short strings spinning and rotating in $AdS_5 \times S^5$. They probe a small curvature region of AdS_5. Moreover, if the boost energy is much smaller than the rotational energy, i.e. $\nu^2 \ll \mathcal{S}$, then

$$\mathcal{E} \approx \sqrt{2\mathcal{S}} + \frac{\nu^2}{2\sqrt{2\mathcal{S}}}\,, \tag{2.17}$$

which indeed is the flat-space Regge trajectory. Besides, when the boost energy is greater than the spin ($2\mathcal{S} \ll \nu$) one obtains

$$E \approx J + S + \frac{\lambda S}{2J^2}\,. \tag{2.18}$$

On the other hand, for long strings, i.e. $\eta \to 0^+$, for $\nu \ll -\log\eta$, it is obtained the relation

$$E \approx S + \frac{\sqrt{\lambda}}{\pi}\log\left(\frac{S}{\sqrt{\lambda}}\right) + \frac{\pi J^2}{2\sqrt{\lambda}\log(S/\sqrt{\lambda})}\,. \tag{2.19}$$

For $\nu = 0$, it becomes the relation that, through the holographic identification $E \equiv \Delta$ (where Δ is the scaling dimension of the corresponding SYM theory operator), leads to the logarithmic anomalous dimension of minimal twist operators.

When $\log(\mathcal{S}/\nu) \ll \nu \ll \mathcal{S}$, we obtain

$$E \approx J + S + \frac{\lambda}{2\pi^2 J}\log^2\left(\frac{S}{J}\right). \tag{2.20}$$

Now, it is also useful to review some issues given in refs. [5, 7], relating results expressed in global and Poincaré coordinates. In Poincaré coordinates the metric of $AdS_5 \times S^5$ space-time is

$$ds^2_{AdS_5\times S^5} = \frac{R^2}{z^2}(dx_m dx_m + dz_p dz_p)\,, \tag{2.21}$$

where $m = 0, \ldots, 3$ and $p = 1, \ldots, 6$, while $z_p z_p = z^2$. Let us consider $z = z(\tau, \sigma)$ as the radial coordinate in the Poincaré parametrization. The transformations between Poincaré and global coordinates of anti de Sitter space-time are the following

$$\begin{aligned}
X_0 &= \frac{x_0}{z} = \cosh\rho\sin t\,, \\
X_i &= \frac{x_i}{z} = n_i\sinh\rho\,, \\
X_4 &= \frac{1}{2z}(-1 + z^2 - x_0^2 + x_i^2) = n_4\sinh\rho\,, \qquad n_i^2 + n_4^2 = 1\,, \\
X_5 &= \frac{1}{2z}(1 + z^2 - x_0^2 + x_i^2) = \cosh\rho\cos t \\
\tan t &= \frac{2x_0}{1 + z^2 - x_0^2 + x_i^2}\,, \\
z &= \frac{1}{\cosh\rho\cos t - n_4\sinh\rho}\,.
\end{aligned} \tag{2.22}$$

The variables X_0, X_i, X_4 and X_5 are the coordinates of $\mathbb{R}^{2,4}$, being the metric of the AdS_5 induced from the flat $\mathbb{R}^{2,4}$ metric by the embedding $X_0^2 + X_5^2 - X_i^2 - X_4^2 = 1$. The unit vector n_k parametrizes the S^3 in AdS_5.

The relation between the energy of AdS_5 expressed in Poincaré and global coordinates is given by

$$\begin{aligned} E &= \sqrt{\lambda}\mathcal{E} = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma \mathcal{E}_d = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma \dot{t}\cosh^2\rho \\ &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma[(1+z^2+x^2)\mathcal{P}_0 - 2x_0\mathcal{D}]\,. \end{aligned} \tag{2.23}$$

Energy density for translations in x_0 is

$$\mathcal{P}_0 = \frac{1}{z^2}\dot{x}_0\,, \tag{2.24}$$

and dilatation charge density is given by

$$\mathcal{D} = \frac{1}{2z^2}\frac{\partial}{\partial\tau}(z^2+x^2)\,, \tag{2.25}$$

with $x^2 = -x_0^2 + x_i^2$. In addition, in the Poincaré patch

$$E = \frac{P_0 + K_0}{2} = \frac{\sqrt{\lambda}}{4\pi}\int d\sigma(\mathcal{P}_0 + \mathcal{K}_0)\,, \tag{2.26}$$

where $\mathcal{K}_0 = (z^2+x^2)\mathcal{P}_0 - 2x_0\mathcal{D}$. The energy density in eq. (2.23) results

$$\mathcal{E}_d = \frac{(1+z^2+x^2)^2}{2z^2}\frac{\partial}{\partial\tau}\left(\frac{x_0}{1+z^2+x^2}\right). \tag{2.27}$$

In particular, for the Klebanov-Tseytlin background a very interesting solution has been obtained [7], derived from a deformation of a point-like string boosted along the circle parametrized by φ on S^5. In Poincaré coordinates we parametrize the undeformed solution as

$$x_0 = \tan t\,, \qquad z = \frac{1}{\cos t}\,, \qquad \varphi = t = \nu\tau\,, \tag{2.28}$$

while, in global coordinates it reads

$$t = \nu\tau\,, \qquad \varphi = \nu\tau\,. \tag{2.29}$$

Similarly, a deformation of a closed string spinning in AdS_5 has been considered [7]. In Poincaré coordinates the undeformed solution is

$$x_0 = \tan t\,, \qquad z = \frac{1}{\cos t\cosh\rho}\,, \qquad x_1 = r\cos\phi\,, \qquad x_2 = r\sin\phi\,, \qquad r = \frac{\tanh\rho}{\cos t}\,, \tag{2.30}$$

and in global coordinates it becomes

$$t = \kappa\tau\,, \qquad \phi = \omega\tau\,, \qquad \rho = \rho(\sigma)\,, \qquad \rho'^2 = \kappa^2\cosh^2\rho - \omega^2\sinh^2\rho\,. \tag{2.31}$$

For this kind of solutions the energy E in global coordinates coincides with the energy in the Poincaré ones, P_0. This is so because $z^2 - x_0^2 + x_i^2 = 1$.

JHEP02(2004)066

In addition, for certain solutions in which we shall be interested, on the side of the type IIA brane constructions we are going to deal with configurations including non-vanishing B-fields. In some cases, this will introduce certain corrections to the spectra of spinning and rotating strings. For example, they will modify some energy-spin and energy-R symmetry charge relation where the rotating strings are stretched along θ_i direction in $T^{1,1}$. So, in this situation θ_i will be a function of σ, and the bosonic lagrangian will be

$$I = I_B - \frac{1}{4\pi\alpha'} \int d\tau d\sigma \epsilon^{ab} B_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \,, \tag{2.32}$$

where ϵ^{ab} is the Levi-Civita tensor density, and $B_{\mu\nu}$ is the Neveu-Schwarz anti-symmetric B-field. However, we shall see that for the configurations studied here the B-field term in the action vanishes. Therefore, the only B-field contributions that will be present in the calculation of energy, spin as well as R-symmetry charge, come from the Nambu-Goto lagrangian, i.e. from the type IIB metric obtained after T and S-duality transformations are applied on the NS5-NS5'-D4 configuration, as we shall see explicitly in the next section. In $AdS_5 \times S^5$ background non-vanishing B-fields configurations have been studied in references [17, 18].

3. Brane constructions in type IIA and type IIB string theories and classical solutions for closed strings

In this section we firstly review the Dasgupta-Mukhi construction which maps a configuration of intersecting branes in type IIA string theory onto a type IIB configuration of N parallel D3-branes on the conifold singularity. Then, inspired on this construction we map classical closed string solutions in $AdS_5 \times T^{1,1}$ background onto classical solutions of strings embedded in configurations of intersecting branes in type IIA string theory.

3.1 Brane constructions and the conifold

The construction proposed by Dasgupta and Mukhi (DM) uses a version of an earlier brane construction developed by Hanany and Witten [43]. DM construction enables one to read off the spectrum and other properties of the conformal field theory on the D3-branes world-volume at the conifold. DM argue that the conifold singularity is represented by a configuration of two type IIA NS 5-branes rotated with respect to each other, and located on a circle S^1, with D4-branes stretched between the NS 5-branes from both sides along S^1.

Let us start by considering a type IIB configuration of a D3-brane stretched between two D5-branes perpendicular to each other, and separated along x^6 (this is the compact direction that parametrizes S^1) as indicated below

$$\begin{array}{rcccccccccc} D5 & 0 & 1 & 2 & 3 & 4 & 5 & - & - & - & - \\ D5' & 0 & 1 & 2 & 3 & - & - & - & - & 8 & 9 \\ D3 & 0 & 1 & 2 & - & - & - & 6 & - & - & - \end{array}$$

In order to obtain an explicit expression for the metric of such a configuration we use the rules given in references [44]–[49], i.e. given M intersecting Dp-branes with harmonic functions H_p, one must choose the maximal set of common directions n_1. The metric for that piece has a common factor $(H_{p_1}H_{p_2}\cdots H_{p_{m_1}})^{-1}$, where m_1 is the number of Dp-branes with n_1 common directions. In the present case n_1 is 3, corresponding to x^0, x^1 and x^2, and leading to the factor $(H_3H_5H_5')^{-1}$ for the piece ds^2_{012}. Then, we must proceed in a similar way with the second set of common directions. In this case, this is given by the direction x^3, including the factor $(H_5H_5')^{-1}$. In this way, the metric becomes

$$\begin{aligned} ds^2 = & (H_3H_5H_5')^{1/2}[(H_3H_5H_5')^{-1}ds^2_{012} + (H_5H_5')^{-1}ds^2_3 + (H_5)^{-1}ds^2_{45} + \\ & + (H_3)^{-1}ds^2_6 + ds^2_7 + (H_5')^{-1}ds^2_{89}] \,, \end{aligned} \tag{3.1}$$

where we see that there is an overall factor. The harmonic functions H_3, H_5 and H_5' only depend on a single overall transverse coordinate x^7. Thus, all H_i behave like $1+|x^7|$. Thus, we have a configuration of partially intersecting branes [45, 46, 50, 41, 51].

Next, we study the metric above under S-duality transformations leading to two perpendicular NS 5-branes as follows

$$\begin{array}{rcccccccccc} NS5 & 0\,1\,2 & 3 & 4 & 5 & - & - & - & - \\ NS5' & 0\,1\,2 & 3 & - & - & - & - & 8 & 9 \\ D3 & 0\,1\,2 & - & - & - & 6 & - & - & - \end{array}$$

being the S-dual metric obtained by multiplying the previous one by an overall factor $(H_5H_5')^{1/2}$

$$\begin{aligned} ds^2 = & (H_3)^{-1/2}ds^2_{012} + (H_3)^{1/2}ds^2_3 + (H_3)^{1/2}H_5'ds^2_{45} + (H_3)^{-1/2}H_5H_5'ds^2_6 + \\ & + (H_3)^{1/2}H_5H_5'ds^2_7 + (H_3)^{1/2}H_5ds^2_{89} \,. \end{aligned} \tag{3.2}$$

Now, we obtain the T-dual version of the metric above in the direction x^3, so that we get a type IIA metric corresponding to D4-branes stretched between the perpendicular NS 5-branes

$$\begin{aligned} ds^2 = & (H_3)^{-1/2}ds^2_{0123} + (H_3)^{1/2}H_5'ds^2_{45} + (H_3)^{-1/2}H_5H_5'ds^2_6 + \\ & + (H_3)^{1/2}H_5H_5'ds^2_7 + (H_3)^{1/2}H_5ds^2_{89} \,, \end{aligned} \tag{3.3}$$

while the intersecting brane configuration is

$$\begin{array}{rcccccccccc} NS5 & 0\,1\,2\,3 & 4 & 5 & - & - & - & - \\ NS5' & 0\,1\,2\,3 & - & - & - & - & 8 & 9 \\ D4 & 0\,1\,2\,3 & - & - & 6 & - & - & - \end{array}$$

At this point, we use the duality map providing the relation between the metric in type IIA string theory, g, and the one in type IIB string theory, G, (see refs. [44, 37]). Notice that on the type IIA string theory configuration non-trivial B-fields can be expected. Thus, there are the following relations

$$G_{mn} = g_{mn} - (g_{6m}g_{6n} - B_{6m}B_{6n})g^{-1}_{66} \,, \qquad G_{66} = g^{-1}_{66} \,, \qquad G_{6m} = B_{6m}g^{-1}_{66} \,, \tag{3.4}$$

where m and n take values from 0 to 9, except 6. In this case we assume non-zero values of B-fields only for the components B_{64} and B_{68}, so that

$$\begin{aligned}
G_{\mu\nu} &= g_{\mu\nu}\,, \qquad \mu,\nu = 0,1,2,3,\\
G_{66} &= g_{66}^{-1}\,,\\
G_{6i} &= B_{6i}g_{66}^{-1}\,, \qquad i = 4,8,\\
G_{ii} &= g_{ii} + B_{6i}^2 g_{66}^{-1}\,,\\
G_{77} &= g_{77}\,,\\
G_{48} &= B_{64}B_{68}g_{66}^{-1}\,.
\end{aligned} \tag{3.5}$$

Therefore, the new type IIB metric is

$$\begin{aligned}
ds^2 = (H_3)^{-1/2}ds_{0123}^2 + (H_3)^{1/2}\Big[&H_5' ds_{45}^2 + H_5H_5' ds_7^2 + H_5 ds_{89}^2 + \\
&+ (H_5H_5')^{-1}(B_{64}ds_4 + B_{68}ds_8 + ds_6)^2\Big]\,.
\end{aligned} \tag{3.6}$$

Following DM we shall argue that this brane configuration is locally a D3-brane at the conifold singularity. This argument is based on the following facts. Firstly, one must consider H_3 to be the harmonic function of a D3-brane localized in the 6 transverse coordinates x_i, $i = 4$ to 9. On the other hand, H_5 and H_5' will continue to be as before $1 + |x^7|$. It means that both 5-brane harmonic functions are non-singular and they approach a constant when $x^7 \to 0$. This allows one to eliminate the 5-brane harmonic functions through a re-scaling of the coordinates (x^4, x^5) and (x^8, x^9), respectively. Finally, it is worth to remark that although the directions (x^4, x^5) and (x^8, x^9) constitute two planes, we need to combine them into the direct product of two 2-spheres with definite radius. This implies an enhancement of global symmetry $\mathrm{U}(1) \times \mathrm{U}(1)$ to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

In order to complete the map we write down certain necessary redefinitions. Let us define $\omega_4 = B_{64}$, and similarly $\omega_8 = B_{68}$, and determine them by solving $\vec{\nabla} \times \vec{\omega} = const.$ By defining $\vec{\omega}_i = (\omega_i, 0, 0)$ for $i = 4$ and 8, and using the *curl* in polar coordinates we get the differential equation

$$\frac{1}{\sin\theta_1}\frac{\partial(\sin\theta_1\omega_4)}{\partial\theta_1} = \mathrm{const} = \tilde{A}_1\,. \tag{3.7}$$

We can solve it obtaining the solution $\sin\theta_1\omega_4 = -\cos\theta_1\tilde{A}_1 + \tilde{A}_2$. Particularly, we can set the constants to be $\tilde{A}_1 = -D$, while $\tilde{A}_2 = 0$. Therefore, we get $\omega_4 = D\cot\theta_1$ and $\omega_8 = D\cot\theta_2$, respectively. In addition, as we have already remarked the fact that the metric (3.6) resembles the structure of D3-branes at the conifold singularity is an encouraging signal in order to attempt to transform that metric in the conifold one. Through the following redefinitions (here we will use the notation $dx^i \equiv ds_i$)

$$ds_4 \to \sqrt{C}\sin\theta_1 d\phi_1\,, \qquad ds_5 \to \sqrt{C}d\theta_1\,, \tag{3.8}$$

and

$$ds_8 \to \sqrt{C}\sin\theta_2 d\phi_2\,, \qquad ds_9 \to \sqrt{C}d\theta_2\,, \tag{3.9}$$

we get the transformation

$$ds_{45}^2 + ds_{89}^2 \to C \sum_{i=1}^{2} (d\theta_i^2 + \sin\theta_i^2 d\phi_i^2)\,. \tag{3.10}$$

In addition, we identify the direction x^6 on the metric (3.6) with the Hopf fiber ψ multiplied by the integration constants $\sqrt{C}D$. We have assumed that x^6 is compactified on S^1 while, on the other hand, the Hopf fiber takes values in the range $[0, 4\pi)$. Including all the mentioned replacements we get

$$(B_{64}ds_4 + B_{68}ds_8 + ds_6)^2 \to CD^2(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,, \tag{3.11}$$

setting $CD^2 = 1/9$ and $C = 1/6$, using the redefinition $x_7 = \log r$ and suitably rescaling coordinates, we get the metric

$$\begin{aligned} ds^2 = {} & (H_3)^{-1/2} ds_{0123}^2 + (H_3)^{1/2}\Bigg[dr^2 + r^2\frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin\theta_i^2 d\phi_i^2) + \\ & + r^2\frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\Bigg]\,, \end{aligned} \tag{3.12}$$

which turns out to be the metric of D3-branes at the conifold singularity.

At this point one interesting question arises, it is whether we can follow the DM construction in a reverse way, i.e. starting from the conifold metric (3.12). An interesting point consists in breaking the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetry down to $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry. In doing this we must guarantee that the base of the conifold $S^2 \times S^3$ will be mapped onto two 2-planes plus a circle which will be related to the compact x^6 coordinate in the brane construction. The first point to notice is that the piece defining the infinitesimal length on any of S^2's

$$d\theta_i^2 + \sin^2\theta_i d\phi_i^2\,, \tag{3.13}$$

for small values of θ_i looks like

$$d\theta_i^2 + \theta_i^2 d\phi_i^2\,. \tag{3.14}$$

This is nothing but the manifestation of the elementary fact that we can, at least locally, break the isometry $\mathrm{SU}(2)$ down to $\mathrm{U}(1)$. So, we now can take θ_i to be a radial variable $\tilde\rho$ (in the sense of polar coordinates), and by extending the metric (3.13) for all positive values of $\tilde\rho$ we get the inversion of the relations (3.8) and (3.9). In the next section we start discussing the rest of transformations in order to get the brane constructions for the specific cases in which we are interested.

3.2 Classical solutions for closed strings in type IIA and type IIB backgrounds

In reference [5] it has been obtained classical closed string solutions rotating around the equatorial circle of S^5, as well as, in the three-sphere parametrizing the anti de Sitter space, and it has been understood how these solutions are related to large R-symmetry charge and large spin operators in the dual $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM theory in 4d, respectively. For our

JHEP02(2004)066

present program we are firstly interested in four kinds of closed string solutions rotating and spinning in the conifold background. The first one corresponds to a closed point-like string orbiting along ψ. Since after undoing the DM construction the Hopf fiber is related to the compact x^6 in the intersecting brane construction, this solution will be mapped onto a point-like closed string orbiting along S^1.

The second solution corresponds to a point-like closed string boosted along θ_i in a 2-cycle of $T^{1,1}$. This solution maps onto a point-like string boosted along the direction x^4 (or x^8) in the type IIA brane setup. The third kind of solution that we study here is a closed string centered in any of the two 2-spheres in $T^{1,1}$, and rotating around its center. When it is transformed onto the corresponding embedding in the intersecting brane construction, this configuration renders two types of equivalent classical closed string solutions moving on the planes (x^4, x^5) and (x^8, x^9), respectively. We also study the case corresponding to a closed string rotating on the S^3 in AdS_5, that can be understood as a closed string rotating in the directions x^1, x^2, x^3 on the type IIA intersecting brane construction.

We start from the near horizon limit of the metric (3.12). Rewriting it global coordinates

$$\begin{aligned} ds^2 = L^2 \Big[&-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3 + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin\theta_i^2 d\phi_i^2) + \\ &+ \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \Big] , \end{aligned} \tag{3.15}$$

where $L^4 = \left(\frac{27}{16}\right) 4\pi g_s N \alpha'^2$.

A point-like closed string boosted along the fiber ψ in $T^{1,1}$. Consider the following parameterization for a string solution embedded in the conifold metric given by

$$\begin{aligned} &t = \kappa\tau , \qquad \psi = \nu\tau , \\ &\rho = 0 , \qquad \theta_i = \phi_i = 0 , \qquad i = 1,2, \qquad \beta_{1,2,3} = 0 , \end{aligned} \tag{3.16}$$

where β_i belongs to Ω_3. Therefore, using the Nambu-Goto lagrangian the relation between energy and U(1) R-symmetry charge is given by

$$E = \frac{9\kappa}{\nu} J . \tag{3.17}$$

Henceforth we take $\sqrt{\lambda} = L^2/\alpha'$, i.e. using the conifold radius instead of the S^5 one. Since we are interested in studying how this $E - J$ relation changes when we map the conifold onto the NS5-NS5'-D4 brane system, it is useful to perform the following identifications

$$\begin{aligned} \frac{\psi}{3} &\equiv \frac{\nu\tau}{3} \equiv x_6 , \\ \sqrt{C}\sin\theta_j d\phi_j &\to dx_i , \qquad j = 1,2, \qquad i = 4,8, \\ \sqrt{C} d\theta_j &\to dx_k , \qquad j = 1,2, \qquad k = 5,9, \end{aligned} \tag{3.18}$$

so that in this way we can write down the metric as seen by the closed point-like string boosted along x^6 direction, i.e. using eq. (3.6)

$$ds^2 = -H_3^{-1/2} dt^2 + H_3^{1/2} (H_5 H_5')^{-1} dx_6^2 . \tag{3.19}$$

This is a type IIB metric related to the conifold one through the map (3.18). Here we have parametrized the corresponding string solution as $t = \kappa\tau$, $x^6 = \nu\tau/3$, and $x^i = 0$ for $i = 1, \ldots, 5$, 8, and 9. We take x^7 to be a constant.

Now, we use the relation between the metric, G, in type IIB string theory and the one in type IIA string theory g, i.e. , $G_{00} = g_{00}, G_{66} = g_{66}^{-1} = H_3^{1/2}(H_5 H_5')^{-1}$. Therefore, the type IIA metric reads

$$ds^2 = -H_3^{-1/2} dt^2 + H_3^{-1/2} H_5 H_5' dx_6^2 \,, \tag{3.20}$$

being the solution a point-like closed string boosted along the compact direction x^6 as above. One can easily calculate the relation

$$E = \frac{1}{H_5 H_5'} \frac{9\kappa}{\nu} J \,. \tag{3.21}$$

Furthermore, one can apply T-duality in direction x^3, and then using S-duality transformations one gets a boosted string along x^6, but in the system of N D3-branes stretched along two perpendicular D5-branes. The energy and J become related again through eq. (3.21), that in the limit for $|x^7| \to 0$ reduces to relation (3.17).

A point-like closed string boosted along θ_i in a 2-cycle of $T^{1,1}$. One can also consider multi-spin solutions as in ref. [12]. In this paper we study the case of a single-folded closed string only spinning in one angular direction. For a string embedded in the conifold metric we use the ansatz

$$\begin{aligned} t &= \kappa\tau \,, & \phi_1 &= \omega_1 \tau \,, & \psi &= 0 \,, \\ \sin\theta_i &= a_i \,, & \cos\theta_i &= b_i \,, & i &= 1, 2 \,, \\ \rho &= 0 \,, & \phi_2 &= \text{constant} \,, & \beta_{1,2,3} &= 0 \,. \end{aligned} \tag{3.22}$$

Therefore, the metric as seen by the string is

$$ds^2 = L^2 \left[-\kappa^2 d\tau^2 + \frac{1}{6} a_1^2 \omega_1^2 d\tau^2 + \frac{1}{9} b_1^2 \omega_1^2 d\tau^2 \right] . \tag{3.23}$$

The Nambu-Goto lagrangian and action read

$$\mathcal{L}_{NG} = L^2 \left(\kappa^2 - \omega_1^2 \left(\frac{a_1^2}{6} + \frac{b_1^2}{9} \right) \right)^{1/2} , \qquad I_{NG} = -\frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \mathcal{L}_{NG} \,. \tag{3.24}$$

Energy and U(1) R-symmetry charge are given by

$$\begin{aligned} E &= -\frac{\partial \mathcal{L}_{NG}}{\partial \kappa} = \kappa \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \frac{1}{\sqrt{\kappa^2 - \omega_1^2 \left(\frac{a_1^2}{6} + \frac{b_1^2}{9} \right)}} , \\ J_1 &= \frac{\partial \mathcal{L}_{NG}}{\partial \omega_1} = \left(\frac{a_1^2}{6} + \frac{b_1^2}{9} \right) \omega_1 \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \frac{1}{\sqrt{\kappa^2 - \omega_1^2 \left(\frac{a_1^2}{6} + \frac{b_1^2}{9} \right)}} , \end{aligned} \tag{3.25}$$

and they are related by

$$E = \frac{\kappa J_1}{\omega_1 \left(\frac{a_1^2}{6} + \frac{b_1^2}{9}\right)}. \tag{3.26}$$

Now, let us identify $\sqrt{C}\sin\theta_1 d\phi_1 \equiv 1/\sqrt{6}a_1\omega_1 d\tau \to dx^4$, where $C = 1/6$, while $x^6 = 0$ and $dx_5 = dx_8 = dx_9 = 0$. Thus, the type IIB metric (3.6) as seen by the string is

$$ds^2 = -H_3^{-1/2}dt^2 + H_3^{1/2}[H_5' + (H_5 H_5')^{-1}B_{64}^2]dx_4^2\,, \tag{3.27}$$

where we see that a non-trivial B-field is turned on. This is $B_{64} = \sqrt{2/3}\cot\theta_1 = \sqrt{2/3}b_1/a_1$. Since we have taken x^6 as a constant in the metric (3.27), the second term in the action (32) vanishes. However, they do contribute to the energy and momentum through the term $\frac{2}{3}\frac{b_1^1}{a_1^2}(H_5 H_5')^{-1}$. The conjugate momentum related to x^4, J_1, corresponds to a point-like string moving along x^4 direction. We can write again the Nambu-Goto lagrangian and get

$$\begin{aligned}
E &= -\frac{\partial \mathcal{L}_{NG}}{\partial \kappa} = \kappa\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{H_3^{-1/2}}{\sqrt{H_3^{-1/2}\kappa^2 - \frac{1}{6}a_1^2\omega_1^2 H_3^{1/2}\left(H_5' + \frac{2}{3}\frac{b_1^1}{a_1^2}(H_5H_5')^{-1}\right)}}, \\
J_1 &= \frac{\partial \mathcal{L}_{NG}}{\partial \omega_1} = \frac{\omega_1 a_1^2}{6}\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{H_3^{1/2}(H_5' + \frac{2}{3}\frac{b_1^1}{a_1^2}(H_5H_5')^{-1})}{\sqrt{H_3^{-1/2}\kappa^2 - \frac{1}{6}a_1^2\omega_1^2 H_3^{1/2}\left(H_5' + \frac{2}{3}\frac{b_1^1}{a_1^2}(H_5H_5')^{-1}\right)}}.
\end{aligned} \tag{3.28}$$

Therefore, we get the relation

$$E = \frac{6\kappa J_1}{a_1^2\omega_1 H_3\left(H_5' + \frac{2}{3}\frac{b_1^1}{a_1^2}(H_5H_5')^{-1}\right)}, \tag{3.29}$$

that reduces to eq. (3.26) when $\theta_1 = \pi/2$ and $r \to 0$. Now, using the transformations from the type IIB to type IIA metrics we obtain the metric as seen by the string embedded in the NS5-NS5'-D4 brane system

$$ds^2 = -H_3^{-1/2}dt^2 + H_3^{1/2}H_5' dx_4^2\,, \tag{3.30}$$

again, one can write the Nambu-Goto lagrangian and derive

$$\begin{aligned}
E &= -\frac{\partial \mathcal{L}_{NG}}{\partial \kappa} = \kappa\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{H_3^{-1/2}}{\sqrt{H_3^{-1/2}\kappa^2 - \frac{1}{6}a_1^2\omega_1^2 H_3^{1/2}H_5'}}\,, \\
J_1 &= \frac{\partial \mathcal{L}_{NG}}{\partial \omega_1} = \frac{\omega_1 a_1^2}{6}\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{H_3^{1/2}H_5'}{\sqrt{H_3^{-1/2}\kappa^2 - \frac{1}{6}a_1^2\omega_1^2 H_3^{1/2}H_5'}}\,.
\end{aligned} \tag{3.31}$$

Thus,

$$E = \frac{6\kappa J_1}{a_1^2\omega_1 H_3 H_5'}\,. \tag{3.32}$$

After using T and S-duality transformations we get the type IIB D5-D5'-D3 system, obtaining again eq. (3.32). This equation coincides with eqs. (3.26) and (3.29) for $\theta_1 = \pi/2$ corresponding to the equatorial rotation in S^2 in $T^{1,1}$.

A spinning closed string in a 2-cycle of $T^{1,1}$. Now we consider a closed string whose center of mass is not moving on $T^{1,1}$ nor in AdS_5, but the string is spinning around some isolated point, and it is stretched in θ_1 (of S^2 in the base of the conifold). Without loss of generality, let us assume that the center of the string is at the North pole. A convenient ansatz is

$$\begin{aligned} &t = e\tau\,, \qquad \phi_1 = e\omega_1\tau\,, \qquad \theta_1 = \theta_1(\sigma)\,, \qquad \rho(\sigma) = \rho(\sigma+\pi)\,, \\ &\phi_2 = \text{constant}\,, \qquad \theta_2 = \text{constant}\,, \qquad \beta_i = 0 \qquad i = 1,2,3\,. \end{aligned} \tag{3.33}$$

The corresponding metric (for the embedding of our string) reads

$$ds^2 = L^2\left[-e^2d\tau^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{9}\cos^2\theta_1 d\phi_1^2\right]. \tag{3.34}$$

Using the above ansatz on the second conformal constraint, we get

$$-L^2e^2 + \frac{L^2}{6}(\theta_{1,\sigma}{}^2 + \sin^2\theta_1 e^2\omega_1^2) + \frac{L^2}{9}\cos^2\theta_1 e^2\omega_1^2 = 0\,. \tag{3.35}$$

Therefore, we obtain the relation

$$\frac{\theta_{1,\sigma}{}^2}{6} = e^2\left[1 - \left(\frac{1}{18}\sin^2\theta_1 + \frac{1}{9}\right)\omega_1^2\right], \tag{3.36}$$

while the Nambu-Goto lagrangian now reads

$$\mathcal{L}_{NG} = -\frac{\sqrt{\lambda}}{2\pi}\frac{e}{\sqrt{6}}\int_0^{2\pi} d\sigma\theta_{1,\sigma}\sqrt{1 - \frac{\omega_1^2}{18}(2+\sin^2\theta_1)}\,, \tag{3.37}$$

where $\sqrt{\lambda} = L^2/\alpha'$. We can calculate the momentum conjugate to θ_1

$$P_1 = \frac{\partial\mathcal{L}_{NG}}{\partial\theta_{1,\sigma}} = -\frac{\sqrt{\lambda}}{2\pi}\frac{e}{\sqrt{6}}\int_0^{2\pi} d\sigma\sqrt{1 - \frac{\omega_1^2}{18}(2+\sin^2\theta_1)}\,, \tag{3.38}$$

thus, for the maximum $\theta_1 = \pi/2$, $dP_1/d\sigma = 0$, it corresponds to $\omega_1^2 = 6$. The energy and R-symmetry charge are given by

$$\begin{aligned} E &= \frac{2\sqrt{\lambda}}{\pi}\frac{e}{\sqrt{6}}\int_0^{\theta_0}\frac{d\theta_1}{\sqrt{1 - \frac{\omega_1^2}{18}(2+\sin^2\theta_1)}}\,, \\ J_1 &= \frac{2\sqrt{\lambda}}{\pi}\frac{\omega_1}{18}\frac{e}{\sqrt{6}}\int_0^{\theta_0} d\theta_1\frac{(2+\sin^2\theta_1)}{\sqrt{1 - \frac{\omega_1^2}{18}(2+\sin^2\theta_1)}}\,, \end{aligned} \tag{3.39}$$

using the maximum value for θ_1 one can easily get

$$\frac{E}{\sqrt{6}} = J_1 + \frac{e}{3\sqrt{3}}\frac{\sqrt{\lambda}}{\pi}\,. \tag{3.40}$$

Now, we consider how this relation transforms in the intersecting branes setup. After using the definitions

$$dx^4 \to \frac{e\omega_1}{\sqrt{6}}\sin\theta_1 d\tau\,, \qquad dx^5 \to \frac{1}{\sqrt{6}}d\theta_1\,, \qquad B_{64} = \sqrt{\frac{2}{3}}\cot\theta_1\,, \tag{3.41}$$

we obtain a type IIB metric with non-trivial B-fields turned on. Since, the B-fields depend on σ through θ_1, one would expect they lead to two kind of contributions to the energy, also modifying the relation $E-J$. One of the contributions should come from the B-field term in the string action, however this is proportional to

$$B_{64}\partial_a x^6 \partial_b x^4 \,, \tag{3.42}$$

which is trivially zero, since in the parametrization above x^6 has been taken to be a constant. In order for the string to be coupled to that B-field, it is necessary that its center of mass be boosted along x^6, i.e. string spins in S^2 and moves along the fiber ψ as well. The actual contribution to energy and J_1 comes directly from the type IIB metric (3.6)

$$\begin{aligned} ds^2 &= -e^2(H_3)^{-1/2}d\tau^2 + (H_3)^{1/2} \times \\ &\quad \times \left[\frac{1}{6}H_5'(e^2\omega_1^2 \sin^2\theta_1 d\tau^2 + d\theta_1^2) + \frac{1}{9}e^2\omega_1^2(H_5H_5')^{-1}\cos^2\theta_1 d\tau^2\right], \end{aligned} \tag{3.43}$$

thus, we have

$$\begin{aligned} E &= \frac{e}{2\pi\alpha'}\int_0^{2\pi}\frac{d\sigma}{\sqrt{6}}\frac{(H_3)^{-1/4}(H_5')^{1/2}\theta_{1,\sigma}}{\sqrt{(H_3)^{-1/2}-(H_3)^{1/2}[\frac{1}{6}H_5'\sin^2\theta_1+\frac{1}{9}(H_5H_5')^{-1}\cos^2\theta_1]\omega_1^2}} \\ J_1 &= \frac{\omega_1 e}{2\pi\alpha'}\int_0^{2\pi}\frac{d\sigma}{\sqrt{6}}\frac{(H_3)^{3/4}(H_5')^{1/2}\theta_{1,\sigma}[\frac{1}{6}H_5'\sin^2\theta_1+\frac{1}{9}(H_5H_5')^{-1}\cos^2\theta_1]}{\sqrt{(H_3)^{-1/2}-(H_3)^{1/2}[\frac{1}{6}H_5'\sin^2\theta_1+\frac{1}{9}(H_5H_5')^{-1}\cos^2\theta_1]\omega_1^2}} \,. \end{aligned} \tag{3.44}$$

In the limit when $x^7 \to 0$ the relation (3.40) is recovered. The corresponding type IIA metric is obtained through the relation between type IIA and type IIB metrics, and it corresponds to the NS5-NS5'-D4 intersecting brane system. The embedding of this spinning closed string solution in this metric is

$$ds^2 = -e^2(H_3)^{-1/2}d\tau^2 + H_5'(H_3)^{1/2}\left[\frac{1}{6}(e^2\omega_1^2\sin^2\theta_1 d\tau^2 + d\theta_1^2)\right], \tag{3.45}$$

which corresponds to a closed string boosted along x^4. After T and S-duality transformations the metric as seen by the spinning string in the D5-D5'-D3 configuration turns out to be

$$ds^2 = -e^2(H_3H_5H_5')^{-1/2}d\tau^2 + (H_5'H_3)^{1/2}(H_5)^{-1/2}\left[\frac{1}{6}(e^2\omega_1^2\sin^2\theta_1 d\tau^2 + d\theta_1^2)\right], \tag{3.46}$$

also corresponding to a closed string boosted parallel to (x^4, x^5) plane. For $\theta_1 = \pi/2$ and $x^7 \to 0$, the $E-J$ relations derived from metric (3.45) and (3.46) become eq. (3.40).

3.3 Spinning closed strings in AdS_5 and boosted along ψ in $T^{1,1}$

Here we consider the situation where a closed string spins centered in S^3 parametrizing AdS_5, and its center of mass moves along ψ in $T^{1,1}$. This is the most interesting situation in terms of analyzing IR and UV limits in the dual $\mathcal{N}=1$ SYM theory. For the embedding in the $AdS_5 \times T^{1,1}$ metric we use the ansatz

$$\begin{aligned} &t = \kappa\tau\,, \qquad \beta_3 \equiv \phi = \omega\tau\,, \qquad \psi \equiv \varphi = \nu\tau\,, \qquad \rho(\sigma) \equiv \rho(\sigma+2\pi) \\ &\phi_i = \text{constant}\,, \qquad \theta_i = \text{constant}\,, \qquad \beta_i = 0 \qquad i = 1,2\,. \end{aligned} \tag{3.47}$$

We start from the conifold metric (3.15) and using conformal constraints it leads to

$$\left(\frac{d\rho}{d\sigma}\right)^2 = \kappa^2 \cosh^2\rho - \omega^2 \sinh^2\rho - \frac{\nu^2}{9}. \tag{3.48}$$

The second derivative is exactly given by eq. (2.6). Notice that for multi-spin string solutions even a more general rotating string ansatz is possible, if for instance we consider the center of mass of the string boosted along $\theta_i = \eta_i\tau$ and $\phi_i = \Omega_i\tau$ for $i = 1,2$. Now, for simplicity we take $\theta_i = \pi/2$, then $\nu^2/9 \to \nu^2/9 + (\Omega_1^2 + \Omega_2^2)/6$, and therefore we can proceed in a similar way as using the ansatz (3.47), but considering several angular momenta. This problem resembles the situations discussed in [12] and it deserves further study in the context of finding new non-BPS sectors in the dual field theory. The integral of eq. (3.48) leads to

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{\rho_0} d\rho \frac{1}{\sqrt{(\kappa^2 - \frac{\nu^2}{9})\cosh^2\rho - (\omega^2 - \frac{\nu^2}{9})\sinh^2\rho}}\,. \tag{3.49}$$

In analogy with the case of $AdS_5 \times S^5$ there are three conserved quantities, energy, spin and R-symmetry charge J,

$$\begin{aligned} E &= \sqrt{\lambda}\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho \equiv \sqrt{\lambda}\mathcal{E}\,, \\ S &= \sqrt{\lambda}\omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2\rho \equiv \sqrt{\lambda}\mathcal{S}\,, \\ J &= \sqrt{\lambda}\frac{\nu}{9} \int_0^{2\pi} \frac{d\sigma}{2\pi} \equiv \sqrt{\lambda}\frac{\nu}{9}\,, \end{aligned} \tag{3.50}$$

thus, we have the relation

$$E = 9\frac{\kappa}{\nu}J + \frac{\kappa}{\omega}S\,. \tag{3.51}$$

Now, one can explicitly calculate the above integrals in terms of hyper-geometric functions. It is useful to write down the energy-spin relations for both, short and long strings. We introduce the parameter $\eta > 0$

$$\coth^2\rho_0 = \frac{\omega^2 - \frac{\nu^2}{9}}{\kappa^2 - \frac{\nu^2}{9}} = 1 + \eta\,, \tag{3.52}$$

where ρ_0 is the maximum value of ρ for a single folded string. Therefore, from eqs. (3.50) we obtain the following relations

$$
\begin{aligned}
\left(\kappa^2 - \frac{\nu^2}{9}\right)^{1/2} &= \frac{1}{\sqrt{\eta}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;-\frac{1}{\eta}\right), \\
\mathcal{E} &= \frac{\kappa}{\sqrt{\eta}\sqrt{\kappa^2-\frac{\nu^2}{9}}}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;-\frac{1}{\eta}\right), \\
\mathcal{S} &= \frac{\omega}{2\eta\sqrt{\eta}\sqrt{\kappa^2-\frac{\nu^2}{9}}}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-\frac{1}{\eta}\right).
\end{aligned}
\tag{3.53}
$$

Now, we analyze the short and long string limits.

Short strings. For $\rho_0 \to 0$ we have $\eta \gg 1$ from above expressions and the definition of η we get the relations

$$
\kappa^2 \approx \frac{\nu^2}{9} + \frac{1}{\eta}, \qquad \omega^2 \approx \kappa^2 + 1 \approx \frac{\nu^2}{9} + 1 + \frac{1}{\eta}, \tag{3.54}
$$

thus $\omega^2 - \kappa^2 \approx 1$, and using the expression for ρ', an approximate short string solution satisfies

$$
\sinh^2\rho \approx \rho^2 \approx \rho_0^2 \sin^2\sigma \approx \frac{1}{\eta}\sin^2\sigma, \tag{3.55}
$$

therefore

$$
\frac{1}{\eta} \approx \frac{2\mathcal{S}}{\sqrt{1+\frac{\nu^2}{9}}} \ll 1. \tag{3.56}
$$

Then, from eq. (3.51) it is obtained a relation similar to eq. (2.15)

$$
\mathcal{E} \approx 9\sqrt{\frac{\nu^2}{9} + \frac{2\mathcal{S}}{\sqrt{1+\frac{\nu^2}{9}}}} + \frac{\sqrt{\frac{\nu^2}{9} + \frac{2\mathcal{S}}{\sqrt{1+\frac{\nu^2}{9}}}}}{\sqrt{1+\frac{\nu^2}{9} + \frac{2\mathcal{S}}{\sqrt{1+\frac{\nu^2}{9}}}}}\mathcal{S}, \tag{3.57}
$$

For $\nu \ll 1$, then $\mathcal{S} \ll 1$ and eq. (3.57) reduces to

$$
E^2 \approx 9J^2 + 81(2\sqrt{\lambda})S. \tag{3.58}
$$

All these expressions become the corresponding ones of spinning strings in S^3 of AdS_5 in $AdS_5 \times S^5$ when we take $\nu^2/9 \to \nu^2$. However, we have to remember that $\sqrt{\lambda}$ corresponds now to L^2/α', where $L^2 = \sqrt{27/16}R^2$. This shows that the scale is set by the conifold instead of S^5. Eq. (3.58) is the limit for short strings in $AdS_5 \times T^{1,1}$. They probe the small curvature region of AdS_5. Moreover, if the boost energy is much smaller than the rotational energy, i.e. $\nu^2 \ll 6\mathcal{S}$, then

$$
\mathcal{E} \approx 9\sqrt{2\mathcal{S}} + \frac{\nu^2}{2\sqrt{2\mathcal{S}}}, \tag{3.59}
$$

which after scaling $\nu^2/9 \to \nu^2$ is the flat-space Regge trajectory, as in the case of $AdS_5 \times S^5$. On the other hand, when the boost energy is greater than the spin ($6\mathcal{S} \ll \nu$)

$$E \approx 3J + S + 3\left(27 - \frac{1}{2}\right)\frac{\lambda S}{J^2}\,. \tag{3.60}$$

This corresponds to a short string spinning slowly around S^3 in AdS_5 and boosted very fast along ψ. It can be related to the leading quantum term in the spectrum of $AdS_5 \times T^{1,1}$, in the frame boosted to the speed of light along the fiber ψ. As in the case of $AdS_5 \times S^5$, the classical energy (3.60) of the spinning string boosted along ψ should also be obtained from the quantum spectrum of string oscillators in that boosted frame. Now, if we are in a sector where $1 \ll S \ll J$, the quantum spectrum reproduces the classical energy. Specifically, the S term is related to the mass term in light-cone string coordinates, in the plane wave background. It is related to the curvature of the anti de Sitter space.

Long strings. For long strings ρ_0 is large, what implies that $\eta \ll 1$ and therefore,

$$\begin{aligned} \kappa^2 &\approx \frac{\nu^2}{9} + \frac{1}{\pi^2}\log^2\left(\frac{1}{\eta}\right), \\ \omega^2 &\approx \frac{\nu^2}{9} + \frac{1}{\pi^2}(1+\eta)\log^2\left(\frac{1}{\eta}\right), \end{aligned} \tag{3.61}$$

and

$$\mathcal{S} \approx \frac{2\omega}{\eta \log \frac{1}{\eta}}. \tag{3.62}$$

Since, as in the case of $AdS_5 \times S^5$ there is no simple relation between energy and spin, we must consider two limits. If $\nu \ll 3\log(1/\eta)$, it is obtained the relation

$$E \approx S + \frac{9\sqrt{\lambda}}{\pi}\log\left(\frac{S}{\sqrt{\lambda}}\right) + \frac{\pi J^2}{2\sqrt{\lambda}\log(S/\sqrt{\lambda})}, \tag{3.63}$$

while if $\log(3\mathcal{S}/\nu) \ll 3\nu \ll \mathcal{S}$, then

$$E \approx 3J + S + \frac{27\lambda}{2\pi^2 J}\log^2\left(\frac{S}{J}\right). \tag{3.64}$$

Spinning strings and brane constructions. Let us consider the following ansatz for a closed string spinning in the (x_1, x_2) plane and boosted along the direction x_6, embedded in the metric (3.6) as follows

$$x_0 = \kappa\tau\,, \qquad \gamma_3 = \omega\tau\,, \qquad x_6 = \frac{\nu}{3}\tau\,, \qquad x^7(\sigma) = x^7(\sigma + 2\pi)\,, \tag{3.65}$$

where γ_3 parametrizes the rotation in (x_1, x_2), and the rest of coordinates are fixed. Also, we consider the string is stretched along x^7.

The type IIB metric (3.6) as seen by the string is given by

$$ds^2 = (H_3)^{-1/2}(-\kappa^2 + \omega^2)d\tau^2 + (H_3)^{1/2}(H_5 H_5')^{-1}\frac{\nu^2}{9}d\tau^2 + H_3^{1/2}(H_5 H_5')dx_7^2\,. \tag{3.66}$$

There is no contributions from B-fields since x_4 and x_8 are constants. Since we are interested in the type IIA NS5-NS5'-D4 brane construction, we write the corresponding type IIA metric as follows

$$ds^2 = (H_3)^{-1/2}(-\kappa^2 + \omega^2)d\tau^2 + (H_3)^{-1/2}(H_5 H_5')\frac{\nu^2}{9}d\tau^2 + H_3^{1/2}(H_5 H_5')dx_7^2 \,. \qquad (3.67)$$

From the second conformal constraint it is obtained

$$\left(\frac{dx_7(\sigma)}{d\sigma}\right)^2 = \frac{1}{H_3 H_5 H_5'}\left(\kappa^2 - \omega^2 - H_5 H_5' \frac{\nu^2}{9}\right), \qquad (3.68)$$

and therefore

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{x_0^7} \frac{dx_7\sqrt{H_3 H_5 H_5'}}{\sqrt{\kappa^2 - \omega^2 - H_5 H_5' \frac{\nu^2}{9}}} \,. \qquad (3.69)$$

We can also derive explicit expressions for the energy, spin and R-symmetry charge

$$\begin{aligned} E &= 4\frac{\kappa}{2\pi\alpha'}\int_0^{x_0^7} \frac{dx_7\sqrt{H_5 H_5'}}{\sqrt{\kappa^2 - \omega^2 - H_5 H_5' \frac{\nu^2}{9}}} \,, \\ J &= 4\frac{\nu}{18\pi\alpha'}\int_0^{x_0^7} \frac{dx_7 (H_5 H_5')^{3/2}}{\sqrt{\kappa^2 - \omega^2 - H_5 H_5' \frac{\nu^2}{9}}} \,, \\ S &= 4\frac{\omega}{2\pi\alpha'}\int_0^{x_0^7} \frac{dx_7\sqrt{H_5 H_5'}}{\sqrt{\kappa^2 - \omega^2 - H_5 H_5' \frac{\nu^2}{9}}} \,. \end{aligned} \qquad (3.70)$$

In all the expressions above x_0^7 is the maximum length of a single folded string.

In addition, we can consider another kind of closed string spinning in the (x_1, x_2) plane, boosted along the direction x_6. In this case we consider the rotation of the string in the (x_1, x_2) plane parametrized in polar coordinates (r, ϕ) and also stretched along r in this plane, but unlike the example above, this is not stretched along x_7.

$$x_0 = \kappa\tau \,, \qquad \phi = \omega\tau \,, \qquad x_6 = \frac{\nu}{3}\tau \,, \qquad r(\sigma) = r(\sigma + 2\pi) \,. \qquad (3.71)$$

The metric of the type IIA NS5-NS5'-D4 brane construction, as seen by the string, becomes

$$ds^2 = (H_3)^{-1/2}[(-\kappa^2 + \omega^2 r^2(\sigma))d\tau^2 + r'^2 d\sigma^2] + (H_3)^{-1/2}(H_5 H_5')\frac{\nu^2}{9}d\tau^2 \,. \qquad (3.72)$$

From the second conformal constraint we get

$$\left(\frac{dr(\sigma)}{d\sigma}\right)^2 = \kappa^2 - H_5 H_5' \frac{\nu^2}{9} - \omega^2 r^2(\sigma) \,, \qquad (3.73)$$

and

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{r_0} \frac{dr}{\sqrt{\kappa^2 - H_5 H_5' \frac{\nu^2}{9} - \omega^2 r^2(\sigma)}} \,. \qquad (3.74)$$

The expressions for the energy, spin and R-symmetry charge are

$$
\begin{aligned}
E &= 4\frac{\kappa(H_3)^{-1/2}}{2\pi\alpha'}\int_0^{r_0}\frac{dr}{\sqrt{\kappa^2 - H_5 H_5' \frac{\nu^2}{9} - \omega^2 r^2(\sigma)}}\,,\\
J &= 4\frac{\nu(H_3)^{-1/2}}{18\pi\alpha'}\int_0^{r_0}\frac{dr(H_5 H_5')}{\sqrt{\kappa^2 - H_5 H_5' \frac{\nu^2}{9} - \omega^2 r^2(\sigma)}}\,,\\
S &= 4\frac{\omega(H_3)^{-1/2}}{2\pi\alpha'}\int_0^{r_0}\frac{dr r^2}{\sqrt{\kappa^2 - H_5 H_5' \frac{\nu^2}{9} - \omega^2 r^2(\sigma)}}\,.
\end{aligned}
\tag{3.75}
$$

For these particular string solutions we have not succeeded in obtaining explicit energy-spin and energy-charge relations as for the previous cases. Probably, the reason could be that our ansatze (3.65) and (3.71) for the string solutions embedded in the intersecting brane backgrounds, do not properly capture the corresponding features for large spin and large R-symmetry charge operators in the dual SYM theory. It would be very interesting to explore more general string solutions embedded in the above brane constructions related to the conifold background, and identifying those solitons with the corresponding dual SYM theory operators.

4. Classical string solutions and $\mathcal{N} = 1$ SYM

We start with a very brief review of the Klebanov-Tseytlin solution [31]. This takes into account the back-reaction due to the presence of M fractional D3-branes in the set up given by N regular D3-branes placed at the conifold singularity, i.e. it includes higher order corrections in M/N. Starting from the type IIB supergravity action the following ansatz for the Einstein frame metric in 10 dimension has been proposed

$$
\begin{aligned}
ds^2 = L^2\Big\{ & e^{-\frac{2}{3}(B+4C)}(du^2 + e^{2A(u)}dx_m dx^m) + \\
& + \frac{1}{9}e^{2B}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \frac{1}{6}e^{2C}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2))\Big\}\,,
\end{aligned}
\tag{4.1}
$$

where $B = B(u)$ and $C = C(u)$. In the limit when these functions vanish the metric of $AdS_5 \times T^{1,1}$ is recovered. The radius L is proportional to $N^{1/4}$. The case studied in [31] assumes the axion $\mathcal{C} = 0$, that trivially makes zero the kinetic terms for the axion and therefore implies that $*F_3 \wedge H_3 = 0$. On the other hand, a RR 3-form flux through the 3-cycle in $T^{1,1}$ is created due to the presence of the fractional branes [30]. This is $F_3 = d\mathcal{C}_2$, and it turns out to be proportional to the closed 3-form constructed in [29].

As it was explained in [31], the same equations can be derived from a 5d action, obtaining a system of first order differential equations with general solutions depending on u. It leads to a holographic RG flow in the dual field theoretical interpretation. Thus, let us assume that above certain UV cut-off u_0 there exists a SCFT, i.e. the total number of fractional branes is zero. One can reach this configuration by placing M fractional anti

D3-branes at u_0. Certainly, it leads to vanishing values for the functions $q = \frac{2}{15}(B + 4C)$ and $f = -\frac{1}{5}(B - C)$ evaluated at u_0, while the coefficient of the Neveu-Schwarz B-field $T(u_0)$ turns out to be a constant. Moreover, after integration of the first order differential equation for f this setting automatically generates a null function $f(u) = 0$. Therefore, it implies that $B = C = 0$ along the entire RG flow toward the IR. This physically means that the internal manifold $T^{1,1}$ is invariant during the whole RG flow, allowing only to change its overall size. Essentially, one can choose the metric to approach the canonical AdS_5 as $u \to u_0$ ($A(u) \to u$), leading to

$$ds_{10}^2 = e^{-5q} du^2 + e^{2A-5q} dx_n dx_n + e^{3q} dS_{T^{1,1}}^2 \,, \tag{4.2}$$

where in general

$$A(u) = A_0 + q(u) + \frac{2}{3} f(u) + \frac{1}{P} T(u) \,, \tag{4.3}$$

while P is proportional to M and $f(u)$ vanishing along the RG flow.

Beyond the near horizon approximation it is possible to construct asymptotically flat solutions. For $P = 0$, the solution describes regular D3-branes at the conifold singularity. On the other hand, if $P \neq 0$ there is a generalization with logarithmic running charge.

Now, let us consider the metric written in the form

$$ds_{10}^2 = s^{-1/2}(r) dx_n dx_n + h^{1/2}(r)(dr^2 + r^2 dS_{T^{1,1}}^2) \,. \tag{4.4}$$

Comparing this metric with the one of eq. (4.2) one gets

$$s(r) = e^{10q(u) - 4A(u)} \,, \qquad e^{6q(u)} = r^4 h(r) \,. \tag{4.5}$$

In addition

$$\frac{dr}{r} = e^{-4q(u)} du \,. \tag{4.6}$$

In order to obtain an explicit solution we start integrating $T' = dT/du = Pe^{-4q}$, leading to

$$T(r) = \tilde{T} + P \log r \,. \tag{4.7}$$

Now, using the expression for q we get

$$e^{6q} = r^4 h(r) = a_0 e^{4\tilde{Q}/P^2 + 4 \log r} + \frac{1}{4}(\tilde{Q} + P^2 \log r) + \frac{1}{16} P^2 \,. \tag{4.8}$$

From this equation it can be obtained an explicit expression for h

$$h(r) = b_0 + \frac{k_0 + P^2 \log r}{4r^4} \,, \tag{4.9}$$

where $b_0 = a_0 e^{4\tilde{Q}/P^2}$ and $k_0 = \tilde{Q} + P^2/4$. The solution for s is $s(r) = h(r)$. Thus, the metric (4.4) becomes

$$ds_{10}^2 = h^{-1/2}(r) dx_n dx_n + h^{1/2}(r) \left(dr^2 + r^2 dS_{T^{1,1}}^2 \right) , \tag{4.10}$$

JHEP02(2004)066

where

$$
\begin{aligned}
h(r) &= b_0 + \frac{k_0 + P^2 \log r}{4r^4}\,, \\
k_0 &= -P^2 \log r_* = Q + P\tilde{T} + \frac{1}{4}P^2\,.
\end{aligned} \tag{4.11}
$$

Therefore, there is a logarithmic RG flow.

Now, let us consider how is the related type IIA brane construction. First of all, we recall that in addition to the N regular D3 branes, there are M fractional D3 branes located at certain value of the radial coordinate at the conifold. On the other hand, as we have seen at the UV limit a set of M-fractional anti D3 branes has been introduced at some radial distance u_0. Thus, it effectively leads to an UV cut-off in the dual gauge field theory, while at this UV cut-off the gauge theory is conformal invariant. For this case, again, in the UV limit the type IIB supergravity background is $AdS_5 \times T^{1,1}$. For the $AdS_5 \times T^{1,1}$, one started with a configuration of two perpendicular D5 branes and N D3 branes stretched between them. Then, performing S-duality one gets two perpendicular NS5 fivebranes with N D3 branes stretched between them, and after T_3 duality one gets the two perpendicular NS5 branes plus N D4 branes stretched between them. In the compact x^6 direction the two NS5 branes are equally separated from both sides. In the DM construction the only overall direction is x^7, and this is identified with the radial direction in the conifold supergravity background, as $x^7 = \log r$. In addition, one has the freedom to include M fractional D4 branes, parallel to the regular ones but separated in x^7 at a distance, let us say, x^7_*. Now this leads to a bending of the perpendicular NS5 branes as a function of x^7. Now, if we introduce M fractional anti-D4 branes at certain x^7_{UV}, the bending of the NS5 branes will be compensated beyond x^7_{UV}, and therefore recovering the conifold (from the type IIB string theory side). At this UV point the results of section 3 are valid.

In addition, in ref. [7] semi-classical solutions of closed strings expanded around the solutions presented in section 2, corresponding to near-conformal backgrounds, have been studied.

In order to explore closed string solutions around the above UV fixed point one can define different ansatze, and use similar calculations as we have done in section 3, to obtain energy-spin and energy-R symmetry charge relations. It would be very interesting to extend these studies to the deep IR limit of the Klebanov-Strassler solution. In this case spinning string solutions has been recently studied [21]. Besides, one very interesting question is whether solitonic string solutions in the deep IR of Klebanov-Strassler background can be mapped onto intersecting brane constructions. In principle, there is not explicit supergravity solutions for these brane constructions. This is in part due to the fact that the presence of the M fractional D4 branes induce a bending on the NS5 branes in the overall perpendicular direction. A step forward in this direction is to consider the case where no fractional D4 branes are present. This case maps onto the deformed conifold [41]. We leave the investigation of classical string solutions in these very interesting backgrounds and also to extend the string theory states/SYM theory operators map for them for a future work.

5. Discussion

We have studied classical solutions of closed strings boosted and spinning in the conifold background, and also have obtained new solitonic string solutions. After applying the Dasgupta-Mukhi map we have analyzed how these solutions transform onto a background of D4-branes stretched between perpendicular NS fivebranes in type IIA string theory. In these cases, the configurations have certain non-vanishing Neveu-Schwarz B-fields which modify the spectra only in certain special cases, and particularly in the case of the type IIB metric, in one of the steps of the DM construction. Then, we have discussed about classical solutions of closed strings in the UV limit of the Klebanov-Tseytlin background. By using a map also inspired in the DM construction we have been able to understand how the logarithmic running of the couplings in the dual SYM theory is seen from the intersecting brane construction viewpoint, and how classical closed string solutions can be embedded in the UV limit of this background. We have also commented on the possibility to extend these studies to the Klebanov-Strassler solution, using as a previous step the Ohta-Yokono construction, relating the deformed conifold and intersecting brane constructions.

A very interesting issue that we have not addressed in this paper concerns to the finding of the corresponding operators in the dual $\mathcal{N}=1$ SYM theory. Some comments are in order. Following the proposal of Gubser, Klebanov and Polyakov, for the case of the conifold solution one can firstly study the case of the dual conformal field theory to the conifold background developed by Klebanov and Witten, i.e. $\mathcal{N}=1$ $\mathrm{SU}(N)\times\mathrm{SU}(N)$ SYM theory, and consider twist two operators $\mathcal{O}_n$. Those are the operators with the lowest conformal dimension, and they are built out of n covariant derivatives. Classically, their conformal dimension is $\Delta_n=n+2$. Thus, they are called twist 2. On the other hand, for these operators it is expected that the leading term in the anomalous dimensions receives a logarithmic correction to all orders in perturbation theory and non-pertubatively as well [52, 53]. Therefore, $\Delta_n=n+2+f(\lambda)\log n+\cdots$. In the expression above there is a functional dependence on the 't Hooft coupling. In ref. [20] it has been discussed how the twist two operators should behave in the UV limit of the Klebanov-Tseytlin solution for the compact S^3 theory [33]. However, the lack of a full non-linear solution for the dual supergravity does not allow one to test the logarithmic behavior.

On the other hand, after transforming the conifold onto the type IIA NS5-NS5'-D4 intersecting brane system, a similar analysis should be expected. In addition, these results should also be valid in the case of the far UV limit of the Klebanov-Tseytlin background. In terms of the analysis developed by Buchel [20] for the IR limit of the Klebanov-Tseytlin background of [33], one could expect analog results along the RG flow in Klebanov-Tseytlin case. However, one must keep in mind that once the theory flows from the conformal point, the identification above between the classical rotating string energy and the conformal dimension is not completely clear. Morever, one should be aware that there are certain potential limitations of the analysis presented here related to the physical interpretation in terms of gauge field theory operators. In this respect we would like to enphasize that the Dasgupta-Mukhi construction provides a local and approximate map and, therefore it

JHEP02(2004)066

would be very interesting to understand the global structure, as well as, the meaning of the string motions in the type IIA NS5-NS5'-D4 intersecting brane system in the corresponding dual gauge theory.

Besides, the NS5-NS5'-D4 brane configurations can be lifted to M-theory descriptions [37, 38]. Essentially, in M-theory the NS5-NS5'-D4 system becomes two perpendicular cylindrical M5 branes plus N M5 branes, which are compactified in the x^{10} direction. It is very interesting to consider a spinning and rotating closed M2 brane embedded in this M-theory background. After the reduction to type IIA string theory, the above system becomes a NS5-NS5'-D4 brane configuration plus a spinning and rotating closed D2 brane embedded in the corresponding background. Then, one can think of taking T duality to obtain a type IIB NS5-NS5'-D3 brane configuration plus a spinning and rotating closed D1 brane. After S-duality, one will be left with D5-D5'-D3 system and a spinning closed F1 brane. Therefore, we can see that other extended objects can also be considered as solitonic solutions in the context of the Gubser, Klebanov, Polyakov's proposal. It would be expected that these solitonic solutions represent different SYM theory operators via the proposed duality. For instance, in the case of dual 3d SYM theories, closed rotating M2 branes in $AdS_4 \times S^7$ [24], as well as, in $Mink_3 \times \mathcal{M}$ [25], where $\mathcal{M}$ is a special holonomy manifold (for instance Spin(7), CY4 fold, Hyper-Kähler manifolds [54]), have been recently considered. One interesting remark is that for special holonomy manifolds, classical rotating closed M2 branes lead to explicit classical energy-spin and energy-charge relations that, if one trust the correspondence beyond the conformal fixed-point, reproduce the logarithmic behaviour of the anomalous dimension of twist two operators in the dual SYM theories. Although, as we mentioned this is related to dual 3d SYM theories, we would expect that a similar behavior can be found for the M-theory descriptions related to the type IIA NS5-NS5'-D4 brane systems that we study in this paper, which are dual to 4d SYM theories.

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